WIS–95/5/Feb.–PH
hep-th/9502118

Fermionic Sum Representations for the Virasoro Characters of the Unitary Superconformal Minimal Models

Ernest Baver and Doron Gepner

Department of Particle Physics
The Weizmann Institute
Rehovot 76100
ISRAEL

Abstract

We present fermionic sum representation for the general Virasoro character of the unitary minimal superconformal series ($N = 1$). Example of the corresponding "finitizated" identities relating corner transfer matrix polynomials with fermionic companions is considered. These identities in the thermodynamic limit lead to the generalized Rogers-Ramanujan identities.

1 Introduction

One of the puzzling features of the two dimensional systems is the appearance of the characters of the fixed point conformal field theories in the expressions for the local state probabilities of the lattice models. Corner transfer matrix method [1] enables to extract information about lattice models at and away of criticality. It turns out that the logarithm of the corner transfer matrix is proportional to the Virasoro generator L_0, *i.e.* their spectra have the same degeneracies and spacing. This is rather striking result, since the Virasoro algebra is connected with the conformal symmetry, which operates only at the critical point, in the continuum limit.

Another closely related mysterious result is the appearance of generalized Rogers-Ramanujan identities (GRR) for the characters of rational conformal field theory. Recently numerous generalized Rogers-Ramanujan identities [2–7] were found/conjectured for the Virasoro characters. These identities relate standard "bosonic" representation with the fermionic-like sums of the form:

$$\sum_{l \quad \textbf{restrictions}} q^{\frac{1}{4}lCl-\frac{1}{2}Al} \prod_a \begin{bmatrix} \frac{1}{2}(l(1-C)+u)_a \\ l_a \end{bmatrix}, \tag{1.1}$$

where some of the components of the vector u may be infinite, C is symmetric matrix, and $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$ are Gaussian polynomials defined by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q)_n}{(q)_m (q)_{n-m}}, \tag{1.2}$$

$$(q)_n = \prod_{j=1}^{n} (1-q^j), \tag{1.3}$$

for integers $n \geq m \geq 0$ and $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right] = 0$ otherwise. The beautiful formula for the general Virasoro character of the minimal unitary models was conjectured in [2]. The natural question which may be asked is: whether a generalization possible for other models?

In this work we generalize the result obtained in [2, 3]. We conjecture the general formula for the Virasoro characters of the unitary superconformal minimal models generated by the coset construction $SU(2)_{k-2} \times SU(2)_2/SU(2)_k$. At the last section we will make few steps towards the proof of the conjectured identities, namely we will formulate the stronger conjecture for the "finitizated" polynomials, which in appropriate limit lead to GRR.

2 General Conjecture

Standard “bosonic” representation for the Virasoro characters of the unitary superconformal minimal models, generated by the coset construction $SU(2)_{k-2} \times SU(2)_2/SU(2)_k$, are given by [8]:

$$\chi^{\mathrm{NS}}_{r,s} = \prod_{n=1}^{\infty} \frac{(1+q^{n-\frac{1}{2}})}{(1-q^n)} \sum_{j=-\infty}^{\infty} q^{\gamma^k_{r,s}(j)} - q^{\beta^k_{r,s}(j)} \qquad r-s = 0 \bmod 2, \quad (2.1)$$

$$\chi^{\mathrm{R}}_{r,s} = \prod_{n=1}^{\infty} \frac{(1+q^{n})}{(1-q^n)} \sum_{j=-\infty}^{\infty} q^{\gamma^k_{r,s}(j)} - q^{\beta^k_{r,s}(j)} \qquad r-s = 1 \bmod 2, \quad (2.2)$$

where

$$\gamma^k_{r,s}(j) = \frac{(2k(k+2)j - r(k+2) + ks)^2 - (r(k+2)-ks)^2}{4k(k+2)}, \quad (2.3)$$

$$\beta^k_{r,s}(j) = \frac{(2k(k+2)j + r(k+2) + ks)^2 - (r(k+2)-ks)^2}{4k(k+2)}, \quad (2.4)$$

$$1 \le r \le k-1, \qquad 1 \le s \le k+1. \quad (2.5)$$

Let us introduce the following notations:

$$S_{k-1}\begin{bmatrix} Q \\ A \end{bmatrix}(u \mid q) = \sum_{\substack{l_1 \in Z_{\ge 0} \\ l_2,..,l_{k-1}+Q}} \frac{q^{\frac{1}{4}lC_{k-1}l - \frac{1}{2}Al}}{(q)_{l_2}} \prod_{a=1,3,..k-1} \begin{bmatrix} \frac{1}{2}(lI_{k-1}+u)_a \\ l_a \end{bmatrix}, \quad (2.6)$$

where $l_2,..,l_{k-1} \in 2Z_{\ge 0}$, $Q \in (Z_2)^{k-2}$ denotes restrictions on summation variables $l_2,..,l_{k-1}$, C_{k-1} is the cartan matrix of A_{k-1} and I_{k-1} is the incidence matrix:

$$I_{k-1} = 2 - C_{k-1}, \quad (2.7)$$

$$(I_{k-1})_{a,b} = \delta_{a,b+1} + \delta_{a,b-1}. \quad (2.8)$$

Let also e_a be unit vector in the a direction *i.e.* $(e_a)_b = \delta_{a,b}$, and set $e_a = 0$ for $a \notin \{1,2,...,k-1\}$.

It was found in ref. [2] that the identity character of the coset models $SU(2)_K \times SU(2)_M / SU(2)_{K+M}$ may be represented in the form (1.1) with $A = 0$, $C = C_{K+M-1}$, and $u_M = \infty$, all other $u_a = 0$, for $a = 1, ..., K+M-1$.

We conjecture* that Virasoro characters of the superconformal minimal models may be represented in the form (2.6) with different characteristics u, Q and A:

$$\chi_{r,s} = \frac{1}{(1+\epsilon^{r-s})} q^{-\frac{1}{8}(s-r-\epsilon^{r-s})(s-r-2+\epsilon^{r-s})} S_{k-1} \begin{bmatrix} Q_{r,s} \\ A_{r,s} \end{bmatrix} (u_{r,s} \mid q), \tag{2.9}$$

$$A_{r,s} = e_{s-1}, \tag{2.10}$$

$$u_{r,s} = e_{s-1} + e_{k+1-r} + \epsilon^{r-s} e_1, \tag{2.11}$$

$$Q_{r,s} = (r-1)\rho + (e_{s-2} + e_{s-4} + ...) + (e_{k+2-r} + e_{k+4-r} + ...), \tag{2.12}$$

where $\rho = e_1 + ... + e_{k-1}$, and

$$\epsilon^{r-s} = \begin{matrix} 0 & r-s = 0 \bmod 2 \\ 1 & r-s = 1 \bmod 2 \end{matrix}. \tag{2.13}$$

Here the addition of components in $Q_{r,s}$ is done modulo 2, and afterward the projection on the plane perpendicular to e_1 is taken, *i.e.* the summation variable l_1 always takes positive integer values. We checked this conjecture by computer for low values of k to high order in q.

Note that the symmetry of the conformal grid $\chi_{r,s} = \chi_{k-r,k+2-s}$ implies new identities relating sums of the form (2.6) with different characteristics.

In closing this section we would like to stress the remarkable similarity between the formula (2.9) and the formula conjectured for the general Virasoro character of the minimal series [2].

*The expression for the identity character $\chi_{1,1}$ conjectured in ref.[2] is recovered from Eqs.(2.9–2.12).

3 Lattice Models and “Finitization” of GRR

The classical way to prove identities (2.9) is to consider first stronger statement, namely the equality of finite polynomials which after taking infinite limit will lead to the generalized Rogers-Ramanujan identities. The “finitizated” version of GRR identities is known for the appropriate characters of the ordinary minimal models. It was conjectured [9], and in some cases proved [9, 10, 11] that the expressions for the local state probabilities (before thermodynamic limit is taken) of the Andrews-Baxter-Forrester models coincide with appropriate finitizated fermionic expressions of the form (1.1). Below we will give the example showing that similar finitizated identities exist for the Virasoro characters of the minimal superconformal series.

Consider lattice models which in the critical region correspond to the coset rational conformal field theory $SU(2)_{k-2} \times SU(2)_2/SU(2)_k$ [12]. The state variables l in these models can take the values $l = 1, 2, ..., k+1$. Two state variables l_1 and l_2 siting on the same bond must obey admissibility condition:

$$l_1 \sim l_2 <=> \begin{array}{c} l_1 - l_2 = -2, 0, 2 \\ l_1 + l_2 = 4, 6, .., 2k \end{array}. \tag{3.1}$$

The ground states are labeled by a pair of states, b and c, such that $b \sim c$. The local state probability is defined as the probability to find the state a at the $l_{0,0}$ site, for the (b, c) ground state,

$$P(a|b,c) = \langle \delta(a, l_{0,0}) \rangle. \tag{3.2}$$

In the regime III local state probability is given in terms of one dimensional configuration sum [12],

$$\Phi(a|b,c) = \lim_{L \to \infty} \Phi_L(a|b,c), \tag{3.3}$$

$$\Phi_L(a|b,c) = \sum_{l_i} q^{\sum_{j=1}^{L} \frac{j}{4} |l_{j+2} - l_j|}, \tag{3.4}$$

where $l_1 = a$, $l_{n+1} = b$, $l_{n+2} = c$, and the sum goes over all admissible sequences $l_1 \sim l_2 \sim ... \sim l_{n+2}$. Here q is related to the temperature-like parameter of the lattice model.

It was shown in ref. [12] that the configuration sum $\Phi(a|b,c)$ is given in terms of branching functions c_{rsa} of this coset rational conformal field theory $SU(2)_{k-2} \times SU(2)_2/SU(2)_k$,

$$\Phi(a|b,c) = q^{\nu} c_{rsa}, \tag{3.5}$$

$$r = \frac{1}{2}(b+c-2), \qquad s = \frac{1}{2}(b-c+4). \tag{3.6}$$

Configuration sum $\Phi_L(a|b,c)$ obeys the following recursion relations:

$$\Phi_L(a|b,c) = \sum_{d \sim b} q^{\frac{L}{4}|d-c|} \Phi_{L-1}(a|d,b), \tag{3.7}$$

where the sum is taken over all d admissible with b. These relations together with boundary condition $\Phi_0(a|b,c) = \delta_{a,b}$ uniquely determines configuration sum $\Phi_L(a|b,c)$.

As an example we will consider the case $k = 4$. Let us introduce the following notations:

$$F_L(1|1,3) = \sum_{\substack{l_1 \\ l_2, l_3 even}} q^{\frac{1}{4} l C_3 l} \begin{bmatrix} \frac{1}{2} l_2 \\ l_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(l_1 + l_3 + L) \\ l_2 \end{bmatrix}, \tag{3.8}$$

$$F_L(1|3,1) = \sum_{\substack{l_1 \\ l_2, l_3 even}} q^{\frac{1}{4} l C_3 l} \begin{bmatrix} \frac{1}{2} l_2 \\ l_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(l_1 + l_3 + L + 1) \\ l_2 \end{bmatrix}, \tag{3.9}$$

$$F_L(1|3,5) = \sum_{\substack{l_1 \\ l_2 even, l_3 odd}} q^{\frac{1}{4} l C_3 l} \begin{bmatrix} \frac{1}{2} l_2 \\ l_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(l_1 + l_3 + L + 1) \\ l_2 \end{bmatrix}, \tag{3.10}$$

$$F_L(1|5,3) = \sum_{\substack{l_1 \\ l_2 even, l_3 odd}} q^{\frac{1}{4} l C_3 l} \begin{bmatrix} \frac{1}{2} l_2 \\ l_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(l_1 + l_3 + L) \\ l_2 \end{bmatrix}. \tag{3.11}$$

Now we may formulate the following conjecture:

$$F_L(1|1,3) = \Phi_L(1|1,3), \qquad F_L(1|3,1) = \Phi_L(1|3,1), \tag{3.12}$$

$$F_L(1|3,5) = \Phi_L(1|3,5), \qquad F_L(1|5,3) = \Phi_L(1|5,3), \tag{3.13}$$

where $\Phi_L(a|b,c)$ are defined from Eq. (3.4). In order to prove this statement we have to show that $F_L(a|b,c)$ obey recursion relation (3.7) together with the initial condition.

It is easy to show that $F_L(a|b,c)$ defined above satisfy the same initial condition as $\Phi_L(a|b,c)$, namely:

$$F_0(1|1,3) = \Phi_0(1|1,3) = 1, \tag{3.14}$$

$$F_0(1|3,1) = \Phi_0(1|3,1) = 0, \tag{3.15}$$

$$F_0(1|5,3) = \Phi_0(1|5,3) = 0, \tag{3.16}$$

$$F_0(1|3,5) = \Phi_0(1|3,5) = 0. \tag{3.16}$$

Let us rewrite the equations (3.7) relevant for our case in explicit form:

$$\Phi_L(1|3,3) = \Phi_{L-1}(1|3,3) + \Phi_{L-1}(1|1,3)q^{\frac{L}{2}} + \Phi_{L-1}(1|5,3)q^{\frac{L}{2}}, \tag{3.17}$$

$$\Phi_L(1|3,1) = \Phi_{L-1}(1|1,3) + \Phi_{L-1}(1|3,3)q^{\frac{L}{2}} + \Phi_{L-1}(1|5,3)q^{L}, \tag{3.18}$$

$$\Phi_L(1|3,5) = \Phi_{L-1}(1|5,3) + \Phi_{L-1}(1|3,3)q^{\frac{L}{2}} + \Phi_{L-1}(1|1,3)q^{L}, \tag{3.19}$$

$$\Phi_L(1|1,3) = \Phi_{L-1}(1|3,1), \tag{3.20}$$

$$\Phi_L(1|5,3) = \Phi_{L-1}(1|3,5). \tag{3.21}$$

It is obvious that $F_L(a|b,c)$ defined above Eqs. (3.8-3.11) satisfy equations (3.20) and (3.21). It was checked for numerous values of parameter L that finitizated fermionic expressions $F_L(a|b,c)$ Eqs. (3.8-3.11) do coincide with appropriate $\Phi_L(a|b,c)$.

Taking limit $L \to \infty$ in equations (3.12-3.13) we obtain GRR identities for the branching functions of the coset $SU(2)_{k-2} \times SU(2)_2/SU(2)_k$. For example:

$$c_{111} = \lim_{L\to\infty} F_L(1|1,3) = \sum_{l_1,l_2,l_3 even} \frac{q^{\frac{1}{4}lC_3l}}{(q)_{l_2}} \begin{bmatrix} \frac{1}{2}l_2 \\ l_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}l_2 \\ l_3 \end{bmatrix}, \tag{3.22}$$

$$c_{131} = \lim_{L\to\infty} F_L(1|3,1) = \sum_{\substack{l_1 odd, \\ l_2,l_3 even}} \frac{q^{\frac{1}{4}lC_3l}}{(q)_{l_2}} \begin{bmatrix} \frac{1}{2}l_2 \\ l_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}l_2 \\ l_3 \end{bmatrix}, \tag{3.23}$$

where the limit is taken for L even and we used

$$\lim_{L\to\infty}\begin{bmatrix} L+a \\ n \end{bmatrix} = \frac{1}{(q)_n}. \tag{3.24}$$

Similarly we have:

$$c_{311} = \lim_{L\to\infty} F_L(1|3,5) = \sum_{\substack{l_1,l_2 even \\ l_3 odd}} \frac{q^{\frac{1}{4}lC_3l}}{(q)_{l_2}} \begin{bmatrix} \frac{1}{2}l_2 \\ l_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}l_2 \\ l_3 \end{bmatrix}, \tag{3.25}$$

$$c_{331} = \lim_{L\to\infty} F_L(1|5,3) = \sum_{\substack{l_2 even \\ l_1,l_3 odd}} \frac{q^{\frac{1}{4}lC_3l}}{(q)_{l_2}} \begin{bmatrix} \frac{1}{2}l_2 \\ l_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}l_2 \\ l_3 \end{bmatrix}. \tag{3.26}$$

From these identities immediately follow identities for the Virasoro characters:

$$\chi^{\mathrm{NS}}_{1,1} = c_{111} + c_{131} = \sum_{\substack{l_1 \\ l_2,l_3 even}} \frac{q^{\frac{1}{4}lC_3l}}{(q)_{l_2}} \begin{bmatrix} \frac{1}{2}l_2 \\ l_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}l_2 \\ l_3 \end{bmatrix}, \tag{3.27}$$

$$\chi^{\mathrm{NS}}_{1,5} = c_{311} + c_{331} = \sum_{\substack{l_1 \\ l_2 even, l_3 odd}} \frac{q^{\frac{1}{4}lC_3l}}{(q)_{l_2}} \begin{bmatrix} \frac{1}{2}l_2 \\ l_1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}l_2 \\ l_3 \end{bmatrix}. \tag{3.28}$$

We believe that results presented here may be generalized to other models. In particular similarity between the fermionic sum representations for minimal and superconformal minimal models inspires to look for GRR related to the branching functions/characters of the more general rational coset field theory $SU(2)_K \times SU(2)_M/SU(2)_{M+K}$.

References

[1] G.E. Andrews, R.J. Baxter and P.J. Forrester, *J. Stat. Phys.* 35 (1984) 193.

[2] R. Kedem, T.R. Klassen, B.M. McCoy and E. Melzer, *Phys. Lett.* **307B** (1993) 68.

[3] D. Gepner, Caltech preprint, hep-th/9410033.

[4] A. Berkovich and B. M. McCoy hep-th/9412030; E. Melzer preprint TAUP 2211-94, hep-th/9412154.

[5] O. Foda and Y.-H. Quano hep-th/9407191; O. Foda and Y.-H. Quano hep-th/9408086.

[6] S.O. Warnaar and P.A. Pearce, hep-th/9411009; S.O. Warnaar and P.A. Pearce *J. Phys.* L891(1994).

[7] S. Dasmahapatra, hep-th/9305024; S. Dasmahapatra, hep-th/9404116.

[8] P. Goddard, A. Kent and D. Olive *Comm. Math. Phys.* **103** (1986) 105.

[9] E. Melzer, *Int. J. Mod. Phys.* **A9** (1994)1115.

[10] A. Berkovich *Nucl. Phys.* **B431** (1994) 315.

[11] O. Foda and S.O. Warnaar hep-th/9501088; S.O. Warnaar, hep-th/9501134

[12] E. Date , M. Jimbo, A. Kuniba, T. Miwa and M. Okado *Nucl. Phys.* **B290** (1987) 231.